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Separation of variables in the BC-type Gaudin magnet

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Abstract. The integrable system is introduced based on the Poisson rs-matrix structure. This is a generalization of the Gaudin magnet, and in the $SL(2)$ case is isomorphic to the generalized Neumann model. The separation of variables is discussed for both the classical and quantum cases.

1. Introduction

The classical integrable systems has been formulated in terms of the classical r-matrix [1]. In one sense, the system is proved to be integrable when we can perform the separation of variables; the reduction of a multi-dimensional system to a set of one-dimensional systems (see [2] for a review). Although the separation of variables has been widely known as the Hamiltonian–Jacobi equation, Sklyanin proposed a new technique (functional Bethe ansatz), which is closely related with the (quantum) inverse scattering method. The functional Bethe ansatz method was first applied to a classical top [3], and further applications, to the Toda lattice, Gaudin magnet, and Heisenberg spin chain, have been carried out. This technique is a new tool in studying integrable systems.

We briefly review the separation of variables for the Gaudin magnet [4] in the classical case. The fundamental formulation of integrable system is based on the classical r-matrix structure

$$\{\overset{1}{\mathbf{L}}(u), \overset{2}{\mathbf{L}}(v)\} = [\mathbf{r}(u-v), \overset{1}{\mathbf{L}}(u) + \overset{2}{\mathbf{L}}(v)]. \tag{1.1}$$

Here we have used the standard notation, $\overset{1}{\mathbf{L}}(u) = \mathbf{L}(u) \otimes 1$ and $\overset{2}{\mathbf{L}}(v) = 1 \otimes \mathbf{L}(v)$. As an example of the L-matrix satisfying the linear Poisson structure (1.1), we can take

$$\mathbf{L}(u) = \mathbf{Z} + \sum_{j=1}^{N} \frac{\mathbf{S}_j}{u - z_j}. \tag{1.2}$$

Here $\mathbf{S}_j$ $(j = 1, \ldots, N)$ is a classical $SL(n)$ spin matrix, and its elements S_j^{ab} $(a, b = 1, \ldots, n)$ satisfy the Poisson relation

$$\{S_j^{ab}, S_k^{cd}\} = \delta_{jk} \cdot \left(\delta^{bc}\, S_j^{ad} - \delta^{da}\, S_j^{cb}\right) \tag{1.3}$$

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and $\mathrm{Tr}\,\mathbf{S}_j^2$ is the Casimir element. We suppose that matrix $\mathbf{Z}$ is traceless, $\mathrm{Tr}\,\mathbf{Z}=0$. The L-matrix (1.2) appears as a quasi-classical limit of the inhomogeneous Heisenberg XXX-spin chain [5, 6], and satisfies the linear Poisson relation (1.1) with the classical r-matrix

$$\mathbf{r}(u)=\frac{\mathbf{P}}{u}. \tag{1.4}$$

The matrix $\mathbf{P}$ means a permutation matrix, $P^{ab,cd}=\delta^{ad}\delta^{bc}$, which satisfies $\mathbf{P}\,x\otimes y=y\otimes x$. Note that r-matrix (1.4) is a rational solution of the classical Yang–Baxter equation [7]

$$[\mathbf{r}_{23}(v),\mathbf{r}_{12}(u)]+[\mathbf{r}_{23}(v),\mathbf{r}_{13}(u+v)]+[\mathbf{r}_{13}(u+v),\mathbf{r}_{12}(u)]=0. \tag{1.5}$$

Once the L-matrix satisfies the Poisson structure (1.1), the model can be proved to be integrable in the Liouville sense. If functions of the L-matrix, $\tau_m(u)$, are defined as

$$\tau_m(u)\equiv\frac{1}{m}\,\mathrm{Tr}\,\mathbf{L}(u)^m \tag{1.6}$$

one can see that the $\tau_m(u)$ are spectral invariant, i.e.

$$\{\tau_l(u),\tau_m(v)\}=0. \tag{1.7}$$

The identity (1.7) shows that the $\tau_m(u)$ are generating functions of the constants of motion. The first non-trivial invariant follows from $\tau_2(u)$:

$$\tau_2(u)=\tfrac{1}{2}\,\mathrm{Tr}\,\mathbf{Z}^2+\sum_{j=1}^{N}\frac{H_j}{u-z_j}+\frac{1}{2}\sum_{j=1}^{N}\frac{\mathrm{Tr}\,\mathbf{S}_j\mathbf{S}_j}{(u-z_j)^2} \tag{1.8}$$

where H_j is the Hamiltonian of the $SL(n)$ Gaudin magnet

$$H_j=\mathrm{Tr}\,\mathbf{Z}\mathbf{S}_j+\sum_{k\neq j}^{N}\frac{\mathrm{Tr}\,\mathbf{S}_j\mathbf{S}_k}{z_j-z_k}. \tag{1.9}$$

This model was introduced by Gaudin as an integrable spin system with long-range interaction [8]. Due to the involutiveness of $\tau_m(u)$, one can see that the Hamiltonian of the $SL(n)$ Gaudin magnet is Poisson commutative, i.e.

$$\{H_j,H_k\}=0 \qquad \text{for } j,k=1,\ldots,N. \tag{1.10}$$

The complete integrability of the model in the Liouville sense can be proved directly from (1.7); when we introduce quantities $\tau_{m,j}^{\alpha}$ by

$$\tau_m(u)=\frac{1}{m}\,\mathrm{Tr}\,\mathbf{Z}^m+\sum_{j=1}^{N}\sum_{\alpha=1}^{m}\frac{\tau_{m,j}^{\alpha}}{(u-z_j)^{\alpha}} \tag{1.11}$$

we can see that the quantities $\tau_{m,j}^{\alpha}$ ($m=2,\ldots,N$; $j=1,\ldots,n$; $\alpha=1,\ldots,m-1$) form a commutative family of $Nn(n-1)/2$ independent Hamiltonians.

For this type of the Gaudin magnet (1.9), the separation of variables (functional Bethe ansatz) has been widely studied in both classical and quantum cases [4, 9–14]. Let $\mathcal{A}(\mathbf{L})$ and $\mathcal{B}(\mathbf{L})$ be certain polynomials of degree $n(n-1)/2$ in matrix elements L_{ab}. When we define variables x_j and p_j by

$$\mathcal{B}(\mathbf{L}(x_j))=0 \qquad p_j=\mathcal{A}(\mathbf{L}(x_j)) \tag{1.12}$$

one sees that variables x_j and p_j are canonically Poisson conjugate [15–17]

$$\{x_j,x_k\}=0 \qquad \{p_j,p_k\}=0 \qquad \{p_j,x_k\}=\delta_{jk}. \tag{1.13}$$

This analysis, which is called the separation of variables, makes it possible to calculate the energy spectrum for the quantum Gaudin magnet.

In this way, we can perform the separation of variables for integrable systems formulated in the linear Poisson relation (1.1). Some of integrable systems, e.g. nonlinear integrable equations on finite segment, are formulated in terms of another Poisson structure; there exists an *'rs'-Poisson structure* [18, 19]

$$\{\overset{1}{\mathbf{L}}(u), \overset{2}{\mathbf{L}}(v)\} = [\mathbf{r}(u-v), \overset{1}{\mathbf{L}}(u) + \overset{2}{\mathbf{L}}(v)] + [\mathbf{s}(u+v), -\overset{1}{\mathbf{L}}(u) + \overset{2}{\mathbf{L}}(v)]. \tag{1.14}$$

Here $\mathbf{r}$ and $\mathbf{s}$ are matrix structure constants. We remark that the s-matrix depends on the sum of the spectral parameters, while the r-matrix on their difference. The Poisson structure (1.14) can be viewed as a classical limit of the boundary Yang–Baxter equation [20], which is used to formulate the quantum spin chain with an open boundary.

In this paper we shall study the separation of variables for the BC-type integrable system formulated by the rs-Poisson structure (1.14). In section 2, we introduce the BC-type $SL(n)$ Gaudin magnet. We give classical r- and s-matrices, and prove their integrability. We relate the Hamiltonian of the BC-type $SL(2)$ Gaudin magnet to the generalized Neumann model in section 3. The separation of variables is also studied. In section 4, we turn our attention to the quantum case. The energy spectrum is given based on the separation of variables. Section 5 is devoted to discussions and conclusions.

2. Gaudin magnet with a boundary

The quantum Gaudin magnet, whose Hamiltonian has a form (1.9), was first introduced in [8] as an integrable spin system with long-range interaction, and solved by use of the coordinate Bethe ansatz. As reviewed in section 1, this original Gaudin magnet can be formulated with the linear Poisson structure in classical case (1.1). In this section, we consider the $SL(n)$ Gaudin magnet with boundary (BC-type Gaudin magnet). The dynamical variables of this model are the classical $SL(n)$ spin S_j^{ab} $(a, b = 1, \ldots, n;\ j = 1, \ldots, N)$ satisfying the Poisson bracket (1.3). Consider the modified L-matrix

$$\mathbf{L}(u) = \sum_{j=1}^{N} \left(\frac{\mathbf{S}_j}{u - z_j} + \frac{\overline{\mathbf{S}}_j}{u + z_j} \right) \tag{2.1}$$

where the 'reflected' classical spin $\overline{\mathbf{S}}$ is defined as

$$\overline{S}^{ab} = (-)^{a+b} S^{ab}. \tag{2.2}$$

Note the difference compared with the usual L-operator (1.2). The second term in (2.1) is due to the effect of the reflection; the classical spin $\mathbf{S}_j$ is located at coordinate z_j, while the 'reflected' spin $\overline{\mathbf{S}}_j$ is at $-z_j$. For this reason we say that the system has a 'boundary'. One can easily check that the modified L-operator (2.1) satisfies the linear Poisson structure (1.14) with the rational r-matrix (1.4) and s-matrix

$$\mathbf{s}(u) = \frac{\overline{\mathbf{P}}}{u} \tag{2.3}$$

where we use the notation $\overline{P}^{ab,cd} = (-)^{a+b} \delta^{ad} \delta^{bc}$.

Let us define functions $\tau_m(u)$ of the matrix $\mathbf{L}(u)$ (2.1) as

$$\tau_m(u) = \frac{1}{m} \operatorname{Tr} \mathbf{L}(u)^m. \tag{2.4}$$

From the Poisson structure (1.14), it can be shown that the functions $\tau_m(u)$ are the spectral invariants of the dynamical system

$$\{\tau_l(u), \tau_m(v)\} = 0. \tag{2.5}$$

The first non-trivial invariant is given from $\tau_2(u)$ as

$$\tau_2(u) = \tfrac{1}{2}\,\mathrm{Tr}\,\mathbf{L}(u)^2 = \sum_{j=1}^{N} \frac{2z_j H_j}{(u-z_j)(u+z_j)} + \frac{1}{2}\sum_{j=1}^{N}\left(\frac{1}{(u-z_j)^2} + \frac{1}{(u+z_j)^2}\right)\mathrm{Tr}\,\mathbf{S}_j^2 \tag{2.6}$$

where H_j has the form

$$H_j = \frac{\mathrm{Tr}(\mathbf{S}_j\overline{\mathbf{S}}_j)}{2z_j} + \sum_{k\neq j}^{N}\left(\frac{\mathrm{Tr}(\mathbf{S}_j\mathbf{S}_k)}{z_j - z_k} + \frac{\mathrm{Tr}(\mathbf{S}_j\overline{\mathbf{S}}_k)}{z_j + z_k}\right). \tag{2.7}$$

The involutiveness of the spectral invariants $\tau_m(u)$ indicates the Poisson commutativity of the Hamiltonian H_j

$$\{H_j, H_k\} = 0 \qquad \text{for } j, k = 1, 2, \ldots, N. \tag{2.8}$$

We call H_j the Hamiltonian of the $SL(n)$ Gaudin magnet with a boundary. Different from the original Gaudin magnet (1.9), the Hamiltonian H_j includes interaction terms between classical spins $\mathbf{S}_j$ and 'reflected' spins $\overline{\mathbf{S}}_j$. By calculating the residues of the $\tau_m(u)$ (2.4) as in the case of original Gaudin magnet (1.11), one can get the 'higher-order' Hamiltonian $\tau^{\alpha}_{m,j}$ of the Gaudin magnet. The existence of a commutative family $\{\tau^{\alpha}_{m,j}\}$ supports the complete integrability of the model in the Liouville sense.

3. Separation of variables

In this section we first show the isomorphism of the N-site $SL(2)$ Gaudin magnet and the N-dimensional generalized Neumann model, and then study the separation of variables based on the technique of Sklyanin. In $SL(2)$ case we can define the classical spin matrices $\mathbf{S}_j$ and $\overline{\mathbf{S}}_j$ in the L-operator (2.1) as

$$\mathbf{S}_j = \begin{pmatrix} S_j^z & S_j^- \\ S_j^+ & -S_j^z \end{pmatrix} \qquad \overline{\mathbf{S}}_j = \sigma^z \mathbf{S}_j \sigma^z \tag{3.1}$$

where σ^z denotes the Pauli spin matrix. These spin variables satisfy the following Poisson relations:

$$\{S_j^z, S_k^{\pm}\} = \pm\delta_{jk}\, S_j^{\pm} \qquad \{S_j^+, S_k^-\} = 2\delta_{jk}\, S_j^z. \tag{3.2}$$

The above Poisson structures for spin variables can be realized with new variables x_j and p_j as

$$S_j^+ = \tfrac{1}{2}\mathrm{i}x_j^2 \qquad S_j^- = \tfrac{1}{2}\mathrm{i}p_j^2 \qquad S_j^z = -\tfrac{1}{2}x_j p_j \tag{3.3}$$

where $\{x_j, p_j | j = 1, \ldots, N\}$ are canonical variables satisfying the Poisson relations $\{x_j, x_k\} = \{p_j, p_k\} = 0$, and $\{x_j, p_k\} = \delta_{jk}$. In this case the Casimir element is set to be zero, $\mathrm{Tr}\,\mathbf{S}_j^2 = 0$. In terms of canonical variables, one can obtain the Hamiltonian from the spectral invariant $\tau_2(u)$ as

$$\tau_2(u) = \tfrac{1}{2}\,\mathrm{Tr}\,\mathbf{L}(u)^2 = \sum_{j=1}^{N} \frac{z_j}{(u-z_j)(u+z_j)} H_j \tag{3.4}$$

where H_j is calculated to be

$$H_j = \frac{x_j^2 p_j^2}{z_j} - \frac{1}{2}\sum_{k\neq j}^{N}\left(\frac{(p_j x_k - x_j p_k)^2}{z_j - z_k} - \frac{(p_j x_k + x_j p_k)^2}{z_j + z_k}\right). \tag{3.5}$$

This Hamiltonian can be viewed as a generalization of the Neumann model [9, 11]. This proves the fact that the N-dimensional generalized Neumann model (3.5) is isomorphic to the N-site BC-type $SL(2)$ Gaudin magnet (2.7).

Now we study the separation of variables for the generalized Neumann model (3.5). Define $A(u)$ and $B(u)$ as

$$A(u) = L_{11}(u) \qquad B(u) = L_{12}(u). \tag{3.6}$$

The linear Poisson structure (1.14) includes relations between functions $A(u)$ and $B(u)$;

$$\begin{aligned} &\{A(u), A(v)\} = 0 \qquad \{B(u), B(v)\} = 0 \\ &\{A(u), B(v)\} = \frac{2u}{(u-v)(u+v)}\Big(B(u) - B(v)\Big). \end{aligned} \tag{3.7}$$

Both the matrix elements $A(u)$ and $B(u)$ Poisson-commute among themselves.

We choose separable coordinates as zeros of the off-diagonal element of the L-operator

$$B(\pm u_\alpha) = 0 \qquad \text{for } \alpha = 1, 2, \ldots, N-1. \tag{3.8}$$

Note that $u = -u_\alpha$ is the solution if $u = u_\alpha$ solves the equation $B(u) = 0$. By use of a set of variables u_α, we further introduce the canonical variables by

$$v_\alpha \equiv A(u_\alpha). \tag{3.9}$$

From the Poisson relations (3.7) one can see that the variables u_α and v_α are canonically conjugate, i.e.

$$\{u_\alpha, u_\beta\} = 0 \qquad \{v_\alpha, v_\beta\} = 0 \qquad \{u_\alpha, v_\beta\} = \delta_{\alpha\beta}. \tag{3.10}$$

The first two Poisson relations can be proved straightforwardly. The third relation follows from:

$$\{u_\alpha, v_\beta\} = \lim_{u\to u_\alpha} \frac{1}{B'(u)}\left(\frac{1}{u_\beta - u} + \frac{1}{u_\beta + u}\right) B(u).$$

The Poisson relations (3.10) show that the u_α and v_α are canonically conjugate variables, and that the generalized Neumann model (3.5) is separated by transforming the dynamical variables as

$$\{x_j, p_j | j = 1, \ldots, N\} \to \{u_\alpha, v_\alpha | j = 1, \ldots, N-1\}.$$

With the variables u_α and v_α, the action W of the generalized Neumann model (3.5) is written in separated form as

$$W = \sum_{\alpha=1}^{N-1} \int v_\alpha \mathrm{d}u_\alpha. \tag{3.11}$$

The separated variables u_α and v_α can be written explicitly in terms of the x_j and p_j. By definition (3.8) we can solve the coordinates p_j as

$$p_j^2 = \frac{z}{z_j} \cdot \frac{\prod_{\alpha=1}^{N-1}(z_j - u_\alpha)(z_j + u_\alpha)}{\prod_{k\neq j}^{N}(z_j - z_k)(z_j + z_k)} \tag{3.12}$$

with $z \equiv \sum_j z_j p_j^2$. The canonically conjugate variables v_α are also solved as

$$v_\alpha \equiv A(u_\alpha) = -\sum_{j=1}^{N} \frac{u_\alpha x_j p_j}{(u_\alpha - z_j)(u_\alpha + z_j)}. \tag{3.13}$$

We remark that the relation between $\{u_\alpha, v_\alpha\}$ and the Hamiltonian H_j is given by the spectral invariants $\tau_2(u = u_\alpha)$ as

$$v_\alpha^2 = \sum_{j=1}^{N} \frac{z_j H_j}{(u_\alpha - z_j)(u_\alpha + z_j)}. \tag{3.14}$$

4. The quantum case

In this section we consider the quantization of the BC-type Gaudin magnet (2.7). For brevity we study the $SL(2)$ case. We set the L-operator as

$$\mathbf{L}(u) = \sum_{j=1}^{N} \left(\frac{\mathbf{S}_j}{u - z_j} + \frac{\overline{\mathbf{S}}_j}{u + z_j} \right) \tag{4.1}$$

where the spin operator $\mathbf{S}_j$ and reflected spin operator $\overline{\mathbf{S}}_j$ are defined as

$$\mathbf{S}_j = \begin{pmatrix} S_j^z & S_j^- \\ S_j^+ & -S_j^z \end{pmatrix} \qquad \overline{\mathbf{S}}_j = \sigma^z \mathbf{S}_j \sigma^z = \begin{pmatrix} S_j^z & -S_j^- \\ -S_j^+ & -S_j^z \end{pmatrix}. \tag{4.2}$$

Here the operators S_j^z and $S_j^\pm$ denote the bases of the $su(2)$ Lie algebra, and they satisfy the commutation relations

$$\begin{aligned} &[S_j^z, S_k^\pm] = \pm S_j^\pm \delta_{jk} \qquad [S_j^+, S_k^-] = 2 S_j^z \delta_{jk} \\ &(S_j^z)^2 + \tfrac{1}{2}(S_j^+ S_j^- + S_j^- S_j^+) = \ell_j(\ell_j + 1) \qquad \ell_j \in \mathbf{Z}_+/2. \end{aligned} \tag{4.3}$$

We have set ℓ_j to be the spin of the jth site. One can check from direct calculations that the L-operator (4.1) satisfies the quantum analogue of the linear Poisson structure (1.14), namely

$$[\overset{1}{\mathbf{L}}(u), \overset{2}{\mathbf{L}}(v)] = [\mathbf{r}(u - v), \overset{1}{\mathbf{L}}(u) + \overset{2}{\mathbf{L}}(v)] + [\mathbf{s}(u + v), -\overset{1}{\mathbf{L}}(u) + \overset{2}{\mathbf{L}}(v)]. \tag{4.4}$$

In this case the constant r- and s-matrices are defined as

$$\mathbf{r}(u) = -\frac{\mathbf{P}}{u} \qquad \mathbf{s}(u) = \overset{1}{\sigma^z} \, \mathbf{r}(u) \, \overset{1}{\sigma^z}. \tag{4.5}$$

The conserved operators are generated in the same way as in the classical case: the trace of the L-operator (2.4). We get the first non-trivial operator from $\hat{\tau}_2(u)$

$$\begin{aligned} \hat{\tau}_2(u) &= \frac{1}{2} \operatorname{Tr} \mathbf{L}(u)^2 \\ &= \sum_{j=1}^{N} \frac{2 z_j}{(u - z_j)(u + z_j)} \hat{H}_j + \sum_{j=1}^{N} \frac{4 z_j^2 \ell_j (\ell_j + 1)}{(u - z_j)^2 (u + z_j)^2}. \end{aligned} \tag{4.6}$$

Here the quantum operator $\hat{H}_j$ is the Hamiltonian of the quantum BC-type Gaudin magnet

$$\hat{H}_j = \frac{(S_j^z)^2}{2 z_j} + \sum_{k \neq j}^{N} \left(\frac{\operatorname{Tr} \mathbf{S}_j \mathbf{S}_k}{z_j - z_k} + \frac{\operatorname{Tr} \mathbf{S}_j \overline{\mathbf{S}}_k}{z_j + z_k} \right). \tag{4.7}$$

From the commutativity of the generating function $\hat{\tau}_2(u)$, we can see that the operators $\hat{H}_j$ are commutative:

$$[\hat{H}_j, \hat{H}_k] = 0 \qquad \text{for } j, k = 1, \ldots, N, \tag{4.8}$$

which proves the quantum integrability of the BC-type Gaudin magnet. Note that the operator $\hat{H}_j$ (4.7) has been appeared in recent studies of the generalized Knizhnik–Zamolodchikov (KZ) equation [21, 22].

The separation of variables for the quantum case can be performed as follows. When we define operators $A(u)$ and $B(u)$ as

$$A(u) = L_{11}(u) \qquad B(u) = L_{12}(u)$$

we obtain from the quantum rs-structure (4.4) that the commutation relations among operators $A(u)$ and $B(u)$ can be written as

$$\begin{aligned} &[A(u), A(v)] = 0 \qquad [B(u), B(v)] = 0 \\ &[A(u), B(v)] = \frac{2u}{(u-v)(u+v)}\Big(B(u) - B(v)\Big). \end{aligned} \tag{4.9}$$

The entire calculation is essentially the same as in the classical case. In the quantum case, we can also introduce the 'canonical operators' u_α and v_α by

$$\begin{aligned} &B(\pm u_\alpha) = 0 \\ &v_\alpha = A(u_\alpha). \end{aligned} \tag{4.10}$$

These operators u_α and v_α satisfy the commutation relations

$$[u_\alpha, u_\beta] = 0 \qquad [v_\alpha, v_\beta] = 0 \qquad [u_\alpha, v_\beta] = \delta_{\alpha\beta}. \tag{4.11}$$

To perform the separation of variables (4.10) for the quantum BC-type Gaudin magnet, we use the realization of the spin operators, S_j^z and $S_j^{\pm}$:

$$S_j^z = -x_j \frac{\partial}{\partial x_j} + \ell_j, \qquad S_j^- = x_j, \qquad S_j^+ = -x_j \frac{\partial^2}{\partial x_j^2} + 2\ell_j \frac{\partial}{\partial x_j}. \tag{4.12}$$

With this realization, the functional equation, $B(\pm u_\alpha) = 0$, does not include differential operator, and can be solved easily. The result is

$$x_j = \frac{z}{2z_j} \frac{\prod_{\alpha=1}^{N-1}(z_j - u_\alpha)(z_j + u_\alpha)}{\prod_{k\neq j}^{N}(z_j - z_k)(z_j + z_k)} \tag{4.13}$$

where we set $z \equiv \sum_k z_k x_k$. When we change the variables from $\{x_j\}$ to $\{u_\alpha\}$, we can see that operator v_α is represented in terms of u_α by

$$v_\alpha = -\frac{\partial}{\partial u_\alpha} + \Lambda(u_\alpha) \tag{4.14}$$

where we use the function $\Lambda(u)$

$$\Lambda(u) = \sum_{j=1}^{N} \frac{2u\,\ell_j}{(u - z_j)(u + z_j)}. \tag{4.15}$$

We remark that identification of operator v_α (4.14) is consistent with the commutation relations (4.11). A set of operators $\{u_\alpha, v_\alpha\}$ is called the separated operator.

We have now completed separating the variables for the quantum $SL(2)$ Gaudin magnet. In the rest of this section, we show that the energy spectrum of the Gaudin magnet can be calculated from the functional Bethe ansatz. By definition of the generating function, $\hat{\tau}_2(u) = \frac{1}{2}\operatorname{Tr}\mathbf{L}(u)^2$, one can see that

$$v_\alpha^2 - \hat{\tau}_2(u_\alpha) = 0 \tag{4.16}$$

which corresponds to (3.14) in the classical case. With the operator realization of v_α obtained in (4.14), we can read off the identity (4.16) as being the differential operator for the separated spectral problem

$$\psi''(u) - 2\Lambda(u)\psi'(u) + \Big(\Lambda^2(u) - \Lambda'(u)\Big)\psi(u) = \tau_2(u)\psi(u) \tag{4.17}$$

where $\tau_2(u)$ is an eigenvalue of the operator $\hat{\tau}_2(u)$ (4.6). This equation can be seen to be a generalized Lamé equation [10]. To solve this second-order differential equation (4.17), we assume that the wavefunction $\psi(u)$ is a polynomial of u, and that the zeros of $\psi(u)$ are denoted as $\pm\lambda_\alpha$ [4]

$$\psi(u) = \prod_{\alpha=1}^{M}(u - \lambda_\alpha)(u + \lambda_\alpha). \tag{4.18}$$

Substituting the wavefunction $\psi(u)$ in the differential equation (4.17), we can see that the eigenvalue E_j of $\hat{H}_j$ is given by

$$E_j = -2\,\chi(z_j)\,\ell_j - \frac{\ell_j}{z_j} + \sum_{k\neq j}^{N} \frac{4\,z_j\,\ell_j\,\ell_k}{(z_j - z_k)(z_j + z_k)} \tag{4.19}$$

where the function $\chi(u)$ is defined from the wavefunction $\psi(u)$ to be

$$\chi(u) = \frac{\mathrm{d}}{\mathrm{d}u}\log\psi(u) = \sum_{\alpha=1}^{M} \frac{2\,u}{(u - \lambda_\alpha)(u + \lambda_\alpha)}. \tag{4.20}$$

Notice that the zeros of the wavefunction $\psi(u)$ should be fixed to satisfy a set of equations

$$\Lambda(\lambda_\alpha) = \sum_{\beta\neq\alpha}^{M} \frac{2\,\lambda_\alpha}{(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha + \lambda_\beta)} + \frac{1}{2\,\lambda_\alpha}, \qquad \text{for } \alpha = 1, \ldots, M, \tag{4.21}$$

which follows from the conditions in cancellation of the residues at $u = \lambda_\alpha$ in (4.17). This equation is a quasi-classical limit of the Bethe ansatz equation for the open-boundary spin chain, and plays a crucial role in the construction of the integral solution for the generalized KZ equation [22].

5. Discussion

In this paper we have introduced the generalized Gaudin magnet. The Hamiltonian is written as

$$H_j = \frac{\operatorname{Tr}(\mathbf{S}_j\overline{\mathbf{S}}_j)}{2\,z_j} + \sum_{k\neq j}^{N}\left(\frac{\operatorname{Tr}(\mathbf{S}_j\mathbf{S}_k)}{z_j - z_k} + \frac{\operatorname{Tr}(\mathbf{S}_j\overline{\mathbf{S}}_k)}{z_j + z_k}\right).$$

This model can be regarded as the Gaudin magnet with a boundary, and be formulated in terms of the 'classical' reflection equation (1.14). As in the case of the original Gaudin magnet, this model has many interesting aspects; in particular for the $SL(2)$ case, the model is proved to be isomorphic to the generalized Neumann model. In both the classical and quantum cases we have performed the separation of variables. In this analysis the eigenvalue problem can be reduced to the second-order differential equation (Lamé equation). We can obtain the so-called quasi-classical Bethe ansatz equation from this differential equation. The XXZ-Gaudin magnet with boundary will be analysed with the same method.

The point of the separation of variables (functional Bethe ansatz) is to take zeros of the wavefunction $\psi(u)$; the zeros may be identified with the 'rapidities' of the spin-wave from the view point of the inverse scattering method. This kind of analysis was used in recent studies of the Asbel–Hofstadter problem [23]. From the viewpoint of q-polynomial theory the study of Askey–Wilson polynomial in terms of quantum rs-structure may be an interesting problem [24].

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